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The wide-spacing approximation applied to multiple scattering and sloshing problems

By D. V. EVANS
Department of Mathematics, University of Bristol, Bristol BS8 1TW, UK

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Linear water-wave theory is used in conjunction with a wide-spacing approximation to develop closed-form expressions for the reflection and transmission coefficients appropriate to a plane wave incident upon any number of identical equally spaced obstacles in two dimensions, and also to derive a real expression from which the sloshing frequencies, which occur when the bodies are bounded by rigid walls, can be determined. In each case the solution is in terms of known properties of radiation problems associated with any one of the bodies in isolation.

1. Introduction

The wide-spacing approximation has proved to be a powerful and surprisingly accurate tool for solving two-dimensional radiation and scattering problems in linear water-wave theory involving more than one body. The method is based on the assumption that the bodies are sufficiently widely spaced that the local field in the vicinity of one body does not influence the others. Thus the wave field at any point is assumed to have arisen from reflection or transmission of waves by adjacent bodies treated in isolation.

The method was first used by Ohkusu (1970) in considering the catamaran problem of two half-immersed circular cylinders. He achieved good agreement with an exact treatment even when the spacing between the cylinders was not large compared with the wavelength. The same conclusion was arrived at by Srokosz & Evans (1979) who provided a full wide-spacing theory for both scattering and radiation problems involving two bodies. They found, by comparison with more accurate results of Evans & Morris (1972), that the method gave good agreement for the reflection of an incident wave by two partially immersed vertical barriers even when the assumption of wide spacing was not valid.

Martin (1985), using the null-field method, shows how the wide-spacing approximation can be recovered when the exact equations are solved in an appropriate asymptotic limit. He concluded that the approximation required both that the wavelength to spacing ratio be small and also that the size of each body be small compared with the spacing.

In a novel application of the approximation Evans & McIver (1987, hereinafter denoted by I) derived a general equation from which the resonant or sloshing frequencies of oscillation of water in a rectangular tank containing an arbitrary obstacle could be determined. The results were compared with a full linearized theory for the case of a thin vertical baffle either piercing the surface or bottom mounted and totally submerged. For the higher modes the agreement was excellent and even for the lowest mode where the approximate theory might be expected to be poor the agreement was within a few percent of the result using the full theory in most cases.

Having established an approximate but accurate tool for predicting sloshing frequencies it is of interest to seek ways of extending the technique to more than one obstacle. Unfortunately, successive application of the wide-spacing approximation rapidly produces unwieldy expressions. Thus the expression given by equation (3.8) in I for the resonant frequencies of water in a tank with three identical equally spaced baffles was only arrived at after lengthy algebra.

Another drawback, not recognized at the time, is that the general equation (3.6) derived in I for the sloshing frequencies in the case of a single body depends upon complex quantities such as the reflection and transmission coefficients for that body in an unbounded region, yet the equation, on physical grounds, must have only real solutions.

Both these shortcomings are overcome in the present paper. First it is shown how, by considering two separate radiation problems for the forced motion of a given body, a real equation can be obtained for the sloshing frequencies for water in a rectangular tank containing an arbitrarily shaped obstacle. For the special case of an obstacle with symmetry about the vertical it is shown how the new result can be manipulated into the form given in I.

Secondly, prompted by ideas of Heading (1982) who considered the related problem of wave propagation through n identical slabs, a closed-form expression is derived for the sloshing frequencies for n identical obstacles in a tank, and for the reflection and transmission coefficients for incident wave transmission past n identical obstacles, in each case in terms of properties of the radiation problem for the forced motion of any one of them. The method, which appears to be more direct than the approach of Heading utilizes a closed-form expression for the nth power of a 2×2 complex matrix of fairly general form.

It is shown that the reflection and transmission coefficients satisfy the conservation of energy flux condition, and the condition under which an incident wave is totally transmitted past all n bodies is derived. This condition turns out to be one of the conditions required for the determination of the resonant frequencies when n bodies are present in the tank, the other condition being the expression for the resonant frequencies for a single obstacle, since by symmetry this will still be satisfied in the more general case.

The power of the technique is illustrated by considering various special cases and by confirming directly the result quoted in I for the resonant frequencies in the case of three thin baffles in a tank.

2. Formulation and solution

We consider two-dimensional motions in the (x,y)-plane. The horizontal bottom is $y=0$ and the undisturbed free surface $y=h$. Linearized water-wave theory is assumed so that we can introduce a velocity potential

$$\Phi(x,y,t) = \mathrm{Re}\,\phi(x,y)\,\mathrm{e}^{-\mathrm{i}\omega t},$$

where ω is the radian frequency. The time-independent complex potential $\phi(x,y)$ satisfies Laplace's equation and

$$K\phi = \partial\phi/\partial y, \quad K=\omega^2/g, \quad y=h, \tag{2.1}$$

$$\partial\phi/\partial y = 0, \quad y=0, \tag{2.2}$$

$\partial\phi/\partial n = 0$ on any obstacles in the fluid, where n is in the direction of the normal to each body.

We assume that the n identical bodies are equally spaced in the interval $0 < x < na$. The mth body $(1 \leqslant m \leqslant n)$ is contained within the mth interval $((m-1)a, ma)$ and contains the point $x = (m-1)a+c = ma-b$ so that $b+c = a$.

It is assumed that at the left-hand boundary of this interval the velocity potential can be written

$$\phi(x,y) \sim (A_{m-1}\,\mathrm{e}^{\mathrm{i}k(x-(m-1)a)} + B_{m-1}\,\mathrm{e}^{-\mathrm{i}k(x-(m-1)a)})\cosh ky, \tag{2.3}$$

and at the right-hand boundary

$$\phi(x,y) \sim (A_m\,\mathrm{e}^{\mathrm{i}k(x-ma)} + B_m\,\mathrm{e}^{-\mathrm{i}k(x-ma)})\cosh ky, \tag{2.4}$$

so that the obstacles are sufficiently widely spaced for any local effects to be neglected. Here k is the positive real root of the equation

$$\omega^2/g \equiv K = k\tanh kh \tag{2.5}$$

so that (2.3) and (2.4) are harmonic and satisfy (2.1) and (2.2).

We shall be primarily concerned with two situations. In the first a wave is incident upon the bodies from either the left of $x = 0$ or the right of $x = na$ and we seek the overall reflection and transmission coefficients. In the second we assume the lines $x = 0$, $x = na$ describe the rigid walls of a rectangular tank and we seek the resonant frequencies ω of free oscillations in the tank in the presence of the obstacles. In either case we shall be prescribing some of the coefficients A_0, B_0, A_n, B_n. In the first case for waves incident upon $x = 0$ we assume

$$\phi \sim (\mathrm{e}^{\mathrm{i}kx} + R_n^{(1)}\,\mathrm{e}^{-\mathrm{i}kx})\cosh ky, \quad x \to -\infty \tag{2.6}$$

$$\sim T_n^{(1)}\,\mathrm{e}^{\mathrm{i}kx}\cosh ky, \quad x \to +\infty, \tag{2.7}$$

so that by comparison with (2.3) and (2.4), $A_0 = 1$, $B_0 = R_n^{(1)}$, $A_n\,\mathrm{e}^{-\mathrm{i}kna} = T_n^{(1)}$, $B_n = 0$, whilst for waves incident upon $x = an$ we assume

$$\phi \sim (\mathrm{e}^{-\mathrm{i}kx} + R_n^{(2)}\,\mathrm{e}^{\mathrm{i}kx})\cosh ky, \quad x \to +\infty \tag{2.8}$$

$$\sim T_n^{(2)}\,\mathrm{e}^{-\mathrm{i}kx}\cosh ky, \quad x \to -\infty, \tag{2.9}$$

so that
$$A_n\,\mathrm{e}^{-\mathrm{i}kna} = R_n^{(2)}, \quad B_n\,\mathrm{e}^{\mathrm{i}kna} = 1, \quad A_0 = 0, \quad B_0 = T_n^{(2)}. \tag{2.10}$$

In the second case in order to satisfy the zero normal velocity condition on the tank walls we require

$$A_0 = B_0, \quad A_n = B_n. \tag{2.11}$$

The aim is to relate the coefficients A_m, B_m to A_{m-1}, B_{m-1} and hence by successive application of this relation, connect A_n, B_n with A_0, B_0. There are a number of ways in which this can be done.

First we set up a new coordinate X in the mth body, where $X = x-(m-1)a-c = x-ma+b$ so that

$$\phi \sim A_{m-1}\,\mathrm{e}^{\mathrm{i}kc}\,\mathrm{e}^{\mathrm{i}kX} + B_{m-1}\,\mathrm{e}^{-\mathrm{i}kc}\,\mathrm{e}^{-\mathrm{i}kX}, \quad X = -c \tag{2.12}$$

$$\sim A_m\,\mathrm{e}^{-\mathrm{i}kb}\,\mathrm{e}^{\mathrm{i}kX} + B_m\,\mathrm{e}^{\mathrm{i}kb}\,\mathrm{e}^{-\mathrm{i}kX}, \quad X = b, \tag{2.13}$$

where here and in what follows we have dropped the cosh ky term for convenience.

Now for each of the bodies in isolation there will be a reflection and transmission coefficient. In particular for the mth body, we assume

$$\phi_1(X,0) \sim e^{ikX} + r_1 e^{-ikX}, \quad X \to -\infty \tag{2.14}$$

$$\sim t_1 e^{ikX}, \quad X \to +\infty, \tag{2.15}$$

and

$$\phi_2(X,0) \sim e^{-ikX} + r_2 e^{ikX}, \quad X \to +\infty \tag{2.16}$$

$$\sim t_2 e^{-ikX}, \quad X \to -\infty, \tag{2.17}$$

where, consistent with our wide-spacing assumption we shall assume that (2.15), (2.16) and (2.14), (2.17) hold at $X = b, -c$ respectively. By considering (2.12) and (2.13) we see that the second term in (2.12) arises from a reflection by the obstacle of the waves described by the first term and a transmission past the obstacle of the waves described by the second term in (2.13). Similarly the first term in (2.13) arises from a reflection by the obstacle of the waves described by the second term and a transmission of the waves described by the first term in (2.12). Thus

$$B_{m-1} e^{-ikc} = A_{m-1} e^{ikc} r_1 + B_m e^{ikb} t_2, \tag{2.18}$$

$$A_m e^{-ikb} = B_m e^{ikb} r_2 + A_{m-1} e^{ikc} t_1, \tag{2.19}$$

which may be written

$$\begin{pmatrix} e^{-ikb} & -r_2 e^{ikb} \\ 0 & t_2 e^{ikb} \end{pmatrix} \begin{pmatrix} A_m \\ B_m \end{pmatrix} = \begin{pmatrix} e^{ikc} t_1 & 0 \\ -r_1 e^{ikc} & e^{-ikc} \end{pmatrix} \begin{pmatrix} A_{m-1} \\ B_{m-1} \end{pmatrix}, \tag{2.20}$$

whence

$$\begin{pmatrix} A_m \\ B_m \end{pmatrix} = \begin{pmatrix} e^{ika}(t_1 - r_1 r_2/t_2) & e^{ikd} r_2/t_2 \\ -e^{-ikd} r_1/t_2 & e^{-ika}/t_2 \end{pmatrix} \begin{pmatrix} A_{m-1} \\ B_{m-1} \end{pmatrix}, \tag{2.21}$$

where $d = b-c$, $a = b+c$.

This is the usual way in which the wide-spacing approximation is applied. If we now consider the resonant frequency problem for a single body so that (2.11) holds with $n(=m) = 1$, then we see from (2.20) that we require

$$t_1 t_2 e^{ika} = (e^{-ikc} - r_1 e^{ikc})(e^{-ikb} - r_2 e^{ikb}) \tag{2.22}$$

which agrees with equation (3.6) of I since it is well known (and will be shown later) that $t_1 = t_2$.

The second approach, adopted by Heading (1982), is to assume

$$\begin{pmatrix} A_m \\ B_m \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} A_{m-1} \\ B_{m-1} \end{pmatrix}, \tag{2.23}$$

and choose possible realizations. For example

$$A_{m-1} e^{ikc} = 1, \quad B_{m-1} e^{-ikc} = r_1, \quad A_m e^{-ikb} = t_1, \quad B_m = 0$$

describes waves incident from the left, whence

$$\begin{pmatrix} t_1 e^{ikb} \\ 0 \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} e^{-ikc} \\ r_1 e^{ikc} \end{pmatrix},$$

whilst $\quad B_m e^{ikb} = 1, \quad A_m e^{-ikb} = r_2, \quad B_{m-1} e^{-ikc} = t_2, \quad A_{m-1} e^{ikc} = 0$

describes waves incident from the right, whence

$$\begin{pmatrix} r_2\,\mathrm{e}^{\mathrm{i}kb} \\ \mathrm{e}^{-\mathrm{i}kb} \end{pmatrix} = \boldsymbol{T}\begin{pmatrix} 0 \\ t_2\,\mathrm{e}^{\mathrm{i}kc} \end{pmatrix}.$$

Putting these together, we get

$$\begin{pmatrix} t_1\,\mathrm{e}^{\mathrm{i}kb} & r_2\,\mathrm{e}^{\mathrm{i}kb} \\ 0 & \mathrm{e}^{-\mathrm{i}kb} \end{pmatrix} = \boldsymbol{T}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}kc} & 0 \\ r_1\,\mathrm{e}^{\mathrm{i}kc} & t_2\,\mathrm{e}^{\mathrm{i}kc} \end{pmatrix},$$

whence

$$\boldsymbol{T} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}ka}(t_1 - r_1 r_2/t_2) & \mathrm{e}^{\mathrm{i}kd}\, r_2/t_2 \\ -\mathrm{e}^{-\mathrm{i}kd}\, r_1/t_2 & \mathrm{e}^{-\mathrm{i}ka}/t_2 \end{pmatrix} \tag{2.24}$$

in agreement with (2.21).

Yet another approach is as follows. Let ψ_i $(i = 1, 2)$ be harmonic potentials satisfying all the conditions of the problem in the mth interval and let

$$\begin{aligned} \psi_i &\sim C_i\,\mathrm{e}^{\mathrm{i}kX} + D_i\,\mathrm{e}^{-\mathrm{i}kX}, \quad X = -c \\ &\sim E_i\,\mathrm{e}^{\mathrm{i}kX} + F_i\,\mathrm{e}^{-\mathrm{i}kX}, \quad X = +b. \end{aligned} \tag{2.25}$$

Then it is well known that, by applying the identity

$$\int_c \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n}\right) \mathrm{d}s = 0,$$

valid for sufficiently smooth harmonic functions, around a contour consisting of the boundary of the mth interval including the free surface, the bottom and the mth body, then

$$C_1 D_2 - C_2 D_1 = E_1 F_2 - E_2 F_1 \tag{2.26}$$

since the only contributions to the integral come from the forms (2.25) at $X = -c$, b. When applied to ψ_1 and $\psi_2 = \bar{\psi}_1$, (2.26) reduces to the condition of energy flux conservation. If we choose $\psi_1 = \phi_1$, $\psi_2 = \phi_2$ and use (2.14)–(2.17) we obtain $t_1 = t_2$ from (2.26). If we choose $\psi_1 = \phi$, as given by (2.12), (2.13) and choose ψ_2 to be first ϕ_1 and then ϕ_2, and apply (2.26) we obtain the equations (2.18), (2.19) with t_1 and t_2 interchanged.

An entirely different result is obtained however by proceeding as follows. For any of the bodies in isolation we can define radiation potentials $\chi_i(X, y)$ $(i = 1, 2)$ resulting from a given *real* normal velocity imposed on the body such as that arising from a rigid body motion of that body. Thus for our mth body in isolation we would have

$$\chi_i(X, 0) \sim A_i^{\pm}\,\mathrm{e}^{\pm\mathrm{i}kX}, \quad X \to \pm\infty,$$

where $A_i^{\pm}$ are assumed known complex potential amplitudes. Consider now

$$\boldsymbol{\Phi}_i = \chi_i - \bar{\chi}_i$$

where a bar denotes complex conjugate. By construction $\boldsymbol{\Phi}_i$ has zero normal velocity on the body. Then

$$\boldsymbol{\Phi}_i(X, 0) \sim A_i^{+}\,\mathrm{e}^{\mathrm{i}kX} - \bar{A}_i^{+}\,\mathrm{e}^{-\mathrm{i}kX}, \quad X \to +\infty \tag{2.27}$$

$$\sim A_i^{-}\,\mathrm{e}^{-\mathrm{i}kX} - \bar{A}_i^{-}\,\mathrm{e}^{\mathrm{i}kX}, \quad X \to -\infty. \tag{2.28}$$

This device was used by Newman (1975) to obtain new relations between radiation and scattering problems – the Newman relations. He chose $\psi_1 = \boldsymbol{\Phi}_i$ and $\psi_2 = \phi_1$ or ϕ_2 and applied (2.26) to the asymptotic forms (2.27), (2.28), and (2.14), (2.15) or

(2.16), (2.17). Here we choose $\psi_1 = \phi$ and $\psi_2 = \Phi_i$, $i = 1, 2$ and, consistent with our previous assumptions, we assume that (2.27), (2.28) hold at b, $-c$ respectively.

It follows from (2.26) that

$$\begin{pmatrix} \bar{A}_1^+ \mathrm{e}^{-\mathrm{i}kb} & A_1^+ \mathrm{e}^{\mathrm{i}kb} \\ \bar{A}_2^+ \mathrm{e}^{-\mathrm{i}kb} & A_2^+ \mathrm{e}^{\mathrm{i}kb} \end{pmatrix} \begin{pmatrix} A_m \\ B_m \end{pmatrix} = \begin{pmatrix} -A_1^- \mathrm{e}^{\mathrm{i}kc} & -\bar{A}_1^- \mathrm{e}^{-\mathrm{i}kc} \\ -A_2^- \mathrm{e}^{\mathrm{i}kc} & -\bar{A}_2^- \mathrm{e}^{-\mathrm{i}kc} \end{pmatrix} \begin{pmatrix} A_{m-1} \\ B_{m-1} \end{pmatrix}. \tag{2.29}$$

If we now apply the resonant conditions (2.11) for a single body we obtain

$$|A_1^-|\,|A_2^+| \cos(kc+\theta_1^-) \cos(kb+\theta_2^+) = |A_1^+|\,|A_2^-| \cos(kb+\theta_1^+) \cos(kc+\theta_2^-), \tag{2.30}$$

where $A_i^\pm = |A_i^\pm| \exp\{\mathrm{i}\theta_i^\pm\}$.

Equation (2.30) is new and, in contrast to (2.22), is a real equation for the real roots k which determine the resonant frequencies from (2.5). Presumably the equivalence of (2.22) and (2.30) may be shown by making use of the Newman relations.

In the special but still interesting case of a body with horizontal symmetry about the line $X = 0$, we have

$$|A_1^+| = |A_1^-| \equiv |A_\mathrm{s}|, \quad \theta_1^+ = \theta_1^- \equiv \theta_\mathrm{s},$$
$$|A_2^+| = |A_2^-| \equiv |A_\mathrm{a}|, \quad \theta_2^+ = \theta_2^- \pm \pi \equiv \theta_\mathrm{a}$$

and (2.29) reduces to

$$\begin{pmatrix} -\mathrm{e}^{\mathrm{i}(kc+\theta_\mathrm{s})} & -\mathrm{e}^{-\mathrm{i}(kc+\theta_\mathrm{s})} \\ \mathrm{e}^{\mathrm{i}(kc+\theta_\mathrm{a})} & \mathrm{e}^{-\mathrm{i}(kc+\theta_\mathrm{a})} \end{pmatrix} \begin{pmatrix} A_{m-1} \\ B_{m-1} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(kb+\theta_\mathrm{s})} & \mathrm{e}^{\mathrm{i}(kb+\theta_\mathrm{s})} \\ \mathrm{e}^{-\mathrm{i}(kb+\theta_\mathrm{a})} & \mathrm{e}^{\mathrm{i}(kb+\theta_\mathrm{a})} \end{pmatrix} \begin{pmatrix} A_m \\ B_m \end{pmatrix}. \tag{2.31}$$

Direct inversion of the matrix on the right-hand side of (2.31) now gives, using (2.23),

$$\boldsymbol{T} = \frac{\mathrm{i}}{\sin B} \begin{pmatrix} -\mathrm{e}^{\mathrm{i}A} & -\mathrm{e}^{\mathrm{i}D} \cos D \\ \mathrm{e}^{-\mathrm{i}D} \cos B & \mathrm{e}^{-\mathrm{i}A} \end{pmatrix}, \tag{2.32}$$

where
$$A = ka + \theta_\mathrm{a} + \theta_\mathrm{s}, \quad D = k(b-c), \quad B = \theta_\mathrm{s} - \theta_\mathrm{a}. \tag{2.33}$$

Now for symmetric bodies the Newman relations reduce to

$$r + t = -\mathrm{e}^{2\mathrm{i}\theta_\mathrm{s}}, \quad r - t = -\mathrm{e}^{2\mathrm{i}\theta_\mathrm{a}}, \tag{2.34}$$

where
$$r_1 = r_2 \equiv r, \quad t_1 = t_2 \equiv t,$$

and substitution of (2.34) into this special case of (2.24) gives (2.32), as expected, after a little algebra. The resonant condition (2.30) for a symmetric body reduces to

$$\cos(kc+\theta_\mathrm{a}) \cos(kb+\theta_\mathrm{s}) + \cos(kc+\theta_\mathrm{s}) \cos(kb+\theta_\mathrm{a}) = 0 \tag{2.35}$$

and with rather more effort it is also possible to prove that this condition is identical to (2.22) in the symmetric case, namely

$$t^2 = (r - \mathrm{e}^{-2\mathrm{i}kb})(r - \mathrm{e}^{-2\mathrm{i}kc}). \tag{2.36}$$

Thus we have from (2.35)

$$\begin{aligned} 0 &= 4\{\cos(kb+\theta_\mathrm{s}) \cos(kc+\theta_\mathrm{a}) + \cos(kc+\theta_\mathrm{s}) \cos(kb+\theta_\mathrm{a})\} \\ &= \mathrm{e}^{\mathrm{i}(kb-\theta_\mathrm{s})} (\mathrm{e}^{-2\mathrm{i}kb} + \mathrm{e}^{2\mathrm{i}\theta_\mathrm{s}}) \mathrm{e}^{\mathrm{i}(kc-\theta_\mathrm{a})} (\mathrm{e}^{-2\mathrm{i}kc} + \mathrm{e}^{2\mathrm{i}\theta_\mathrm{a}}) \\ &\quad + \mathrm{e}^{\mathrm{i}(kc-\theta_\mathrm{s})} (\mathrm{e}^{-2\mathrm{i}kc} + e^{2\mathrm{i}\theta_\mathrm{s}}) \mathrm{e}^{\mathrm{i}(kb-\theta_\mathrm{a})} (\mathrm{e}^{-2\mathrm{i}kb} + e^{2\mathrm{i}\theta_\mathrm{a}}) \\ &= \mathrm{e}^{\mathrm{i}k(b+c)} \mathrm{e}^{-\mathrm{i}(\theta_\mathrm{s}+\theta_\mathrm{a})} \{(\beta - r - t)(\delta - r + t) + (\delta - r - t)(\beta - r + t)\}, \end{aligned}$$

where $\beta = \mathrm{e}^{-2ikb}$, $\delta = \mathrm{e}^{-2ikc}$ and the relations (2.34) have been used. This last expression in curly brackets now reduces to $2\{(r-\beta)(r-\delta)-t^2\}$ in agreement with (2.36).

For the special case of a symmetric obstacle in the middle of a rectangular tank, $b = c$, (2.36) reduces to

$$r \pm t = \mathrm{e}^{-2ikb}$$

or

$$\mathrm{e}^{-2ikb} + \mathrm{e}^{2i\theta_s} = \mathrm{e}^{-2ikb} + \mathrm{e}^{2i\theta_a} = 0$$

from (2.34), whence

$$\mathrm{e}^{-i(kb-\theta_s)} \cos(kb+\theta_s) = \mathrm{e}^{-i(kb-\theta_a)} \cos(kb+\theta_a) = 0$$

in agreement with (2.35) when $b = c$.

In this case we have the simple result

$$\begin{aligned} kb &= -\theta_s + \tfrac{1}{2}\pi(2n-1), \\ &= -\theta_a + \tfrac{1}{2}\pi(2m-1), \quad m, n \text{ integers}, \end{aligned}$$

although it should be remembered that θ_s, θ_a depend on k also, as well as on the dimensions of the obstacle. From now on we shall continue to assume our bodies are symmetrical so that $\boldsymbol{T}$ is given by (2.32).

In order to estimate the effect of all the bodies we need to apply (2.31) repeatedly for $m = 1, 2, \ldots, n$. We find that

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \boldsymbol{T}^n \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \tag{2.37}$$

and the problem is solved if we can find a useful form for $\boldsymbol{T}^n$. This is achieved using the following result for a wide class of 2×2 matrices.

Let
$$\boldsymbol{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det \boldsymbol{T} = 1, \quad ad \neq 1$$

and let
$$\cosh\alpha = \tfrac{1}{2}(a+d), \quad b\lambda\cosh\beta = \tfrac{1}{2}(a-d), \quad \lambda^2 = -c/b. \tag{2.38}$$

Then
$$\boldsymbol{T}_n \equiv \begin{pmatrix} \sinh(\beta+n\alpha) & \lambda^{-1}\sinh n\alpha \\ -\lambda\sinh n\alpha & \sinh(\beta-n\alpha) \end{pmatrix} \Big/ \sinh\beta = \boldsymbol{T}^n. \tag{2.39}$$

The proof follows by induction. We have

$$a = \cosh\alpha + \lambda b\cosh\beta, \quad d = \cosh\alpha - \lambda b\cosh\beta. \tag{2.40}$$

So $ad = \cosh^2\alpha - \lambda^2 b^2 \cosh^2\beta = 1 - \lambda^2 b^2$ since $\det \boldsymbol{T} = 1$ and hence

$$\lambda b = \sinh\alpha / \sinh\beta \tag{2.41}$$

and the positive square root has been chosen.

It is now straightforward, using (2.41) to confirm that $\boldsymbol{T}_1 = \boldsymbol{T}$.

Suppose $\boldsymbol{T}_n = \boldsymbol{T}^n$ for all integers up to n. Direct multiplication gives, as the top left element of $\boldsymbol{T}^{n+1} = \boldsymbol{T}^n \cdot \boldsymbol{T}$, the expression

$$\{\sinh(\beta+n\alpha)\sinh(\beta+\alpha) - \sinh n\alpha \sinh\alpha\}/\sinh^2\beta,$$

which, if the induction argument is to work, should equal the corresponding element of $\boldsymbol{T}_{n+1}$ namely, $\sinh(\beta+(n+1)\alpha)/\sinh\beta$. This follows if

$$\sinh(\beta+n\alpha)\sinh(\beta+\alpha) - \sinh(\beta+(n+1)\alpha)\sinh\beta = \sinh n\alpha \sinh\alpha,$$

which is easily verified by expanding the left-hand side in terms of $\cosh n\alpha$ and $\sinh n\alpha$. The other elements of $\boldsymbol{T}^{n+1}$ can be shown to agree with the corresponding elements of $\boldsymbol{T}_{n+1}$, and since we have verified the case $n=1$, it follows that $\boldsymbol{T}_n$ given by (2.39) is indeed $\boldsymbol{T}^n$ for all positive integers n. Notice that from (2.38) we can replace α or β by $-\alpha$ or $-\beta$ in (2.39) to get a different form for $\boldsymbol{T}_n$ also equal to $\boldsymbol{T}^n$. This corresponds to choosing the other square root in the form for b in (2.41).

If $ad=1$ then, since $\det \boldsymbol{T}=1$, one or both of b, c is zero. If both are zero $\boldsymbol{T}$ is diagonal and $\boldsymbol{T}^n$ is straightforward.

Suppose $c=0$, $b\neq 0$, then it is easily proved by induction that

$$\boldsymbol{T}_n \equiv \begin{pmatrix} e^{n\alpha} & b\sinh n\alpha/\sinh\alpha \\ 0 & e^{-n\alpha} \end{pmatrix} = \boldsymbol{T}^n$$

where $a=e^{\alpha}$, $d=e^{-\alpha}$.

This result also follows from the case $ad\neq 1$ by taking the limits $\beta\to\infty$, $\lambda\to 0$ such that $\lambda b\cosh\beta\to\sinh\alpha$, in (2.39), (2.40).

Now the matrix (2.32) has $\det\boldsymbol{T}=1$ and $ad\neq 1$ as required and so it follows that its nth power is given by (2.39), where from (2.32)

$$\cosh\alpha = \sin A/\sin B, \quad \cosh\beta=\cos A/\cos B, \quad \lambda = e^{-iD}. \tag{2.42}$$

Also, from (2.42)

$$\sinh\beta = i\tan B\sinh\alpha. \tag{2.43}$$

Since A, B are real then $|\sin A/\sin B| \gtrless 1$ according as $|\cos A/\cos B| \lessgtr 1$ so that α and β cannot both be real together. We distinguish between the various cases.

Suppose $|\sin A/\sin B|<1$. Then we write $\alpha = i\alpha'$ and $\cos\alpha' = \sin A/\sin B$ and, if $\cos A/\cos B>1, \cosh\beta = \cos A/\cos B$, whereas if $\cos A/\cos B<-1, \cosh\beta' = -\cos A/\cos B$, where $\beta=\beta'+i\pi$.

If $\sin A/\sin B>1$, then $\alpha>0$ and $\cos\beta' = \cos A/\cos B$, where $\beta = i\beta'$. Finally if $\sin A/\sin B<-1, \cosh\alpha' = -\sin A/\sin B$ and $\cos\beta' = \cos A/\cos B$, where $\alpha=\alpha'+i\pi$, $\beta=i\beta'$.

3. Applications

We consider first the reflection and transmission past n symmetric obstacles. We write

$$\boldsymbol{T}_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

and note from (2.7), (2.10) and (2.31) that

$$\left.\begin{aligned} T_n^{(1)} e^{ikna} &= a_n + b_n R_n^{(1)}, \\ 0 &= c_n + d_n R_n^{(1)}, \end{aligned}\right\} \tag{3.1}$$

whilst

$$\left.\begin{aligned} R_n^{(2)} e^{ikna} &= b_n T_n^{(2)}, \\ e^{-ikna} &= d_n T_n^{(2)}. \end{aligned}\right\} \tag{3.2}$$

It follows from (2.39) and (3.1), (3.2) that

$$T_n^{(1)} = T_n^{(2)} \equiv T_n = e^{-ikna}\sinh\beta/\sinh(\beta-n\alpha) \tag{3.3}$$

since $\det\boldsymbol{T}^n=1$,

$$R_n^{(1)} = \lambda\sinh n\alpha/\sinh(\beta-n\alpha), \tag{3.4}$$

$$R_n^{(2)} = e^{-2ikna}\frac{\sinh n\alpha}{\lambda\sinh(\beta-n\alpha)}. \tag{3.5}$$

Now
$$R_n^{(i)}\bar{R}_n^{(i)}+T_n\bar{T}_n=\frac{(\sinh n\alpha)(\sinh n\bar{\alpha})+\sinh\beta\sinh\bar{\beta}}{\sinh(\beta-n\alpha)\sinh(\bar{\beta}-n\bar{\alpha})}\quad(i=1,2),\tag{3.6}$$

and it can be shown by considering each of the separate cases in turn that the right-hand side of (3.6) is unity confirming that conservation of energy flux is satisfied. As illustration we shall suppose in all that follows that $|\sin A/\sin B|<1, \cos A/\cos B>1$ so that $\alpha=\mathrm{i}\alpha'$. Similar considerations apply to the other cases.

Then the numerator and denominator each reduce to

$$\sinh^2\beta+\sin^2 n\alpha',\quad\text{and}\quad |T_n|=|\sinh\beta|/(\sinh^2\beta+\sin^2 n\alpha')^{\frac{1}{2}},\tag{3.7}$$

whilst
$$|R_n^{(i)}|=|\sin n\alpha'|/(\sinh^2\beta+\sin^2 n\alpha')^{\frac{1}{2}}.\tag{3.8}$$

As a check on the correctness of these results we put $n=1$ in (3.7), (3.8) and obtain for a single obstacle, after simple reductions,

$$|R_1^{(i)}|=|\sin\alpha'|/(\cosh^2\beta-\cos^2\alpha')^{\frac{1}{2}}=|\cos B|,\quad |T_i|=|\sin B|$$

where (2.40) with $\alpha=\mathrm{i}\alpha$ has been used.

We note that these expressions are independent of a as expected. But from (2.34)

$$r=-\tfrac{1}{2}(\mathrm{e}^{2\mathrm{i}\theta_s}+\mathrm{e}^{2\mathrm{i}\theta_a})=-\mathrm{e}^{\mathrm{i}(\theta_s+\theta_a)}\cos(\theta_s-\theta_a),$$

whilst
$$t=-\tfrac{1}{2}(\mathrm{e}^{2\mathrm{i}\theta_s}-\mathrm{e}^{2\mathrm{i}\theta_a})=-\mathrm{e}^{\mathrm{i}(\theta_s+\theta_a)}\,\mathrm{i}\sin(\theta_s-\theta_a),$$

whence $|R_1^{(i)}|=|r|=|\cos B|$, $|T_1|=|t|=|\sin B|$ as expected.

Notice that $|R_n^{(i)}|, |T_n|$ depend only on α',β or A,B and not on λ or D showing that the reflected and transmitted amplitudes depend only upon the spacing $a(=b+c)$ and not on b,c separately. This is not the case however for the phases of $R_n^{(i)}, T_n$.

It follows from (3.7) that under the wide-spacing approximation $|T_n|$ never vanishes unless $|T_1|$ vanishes for a single obstacle. The author is not aware of any cases when this occurs. This is in contrast to what is known to occur when the full linearized water-wave equations are used without approximation. Thus Evans & Morris (1972) analytically, and McIver (1985) numerically, have shown that T_2 vanishes at certain discrete wave frequencies in the case of two thin vertical surface barriers in infinite and finite depth respectively.

In contrast it can be seen from (3.8) that $|R_n^{(i)}|=0$ when $\alpha'=m\pi/n, m$ an integer provided $\beta\neq0$ or, from (2.42), whenever

$$\cos m\pi/n=\sin A/\sin B,\quad m\text{ integer},\quad \beta\neq0.\tag{3.9}$$

We consider first the case of a single obstacle. Since $\sinh\beta=-\sin\alpha'\tan B$, both α' and β vanish simultaneously and the appropriate condition for $|R_1^{(i)}|$ to vanish is $|\cos B|=0$. In general this will only occur at discrete wave frequencies but it is noteworthy that for a submerged circular cylinder in deep water $|R_1^{(i)}|=0$ for all wave frequencies and size and depth of submergence of the cylinder (Ursell 1950). It turns out that $|\theta_s-\theta_a|=\frac{1}{2}\pi$ in this case (Ogilvie 1963) so that $\cos B$ vanishes identically.

In the general case $n>1$ we must exclude therefore values of α' satisfying $\sin n\alpha'=0$ which also satisfy $\sin\alpha'=0$ since these belong to the $n=1$ case which has already been considered. Thus there are in general $n-1$ equations to be satisfied for $|R_n^{(i)}|$ to vanish, obtained by choosing $m=1,2,\ldots,n-1$ in (3.9).

It is not difficult to show that for any set of n symmetric obstacles, there exist $n-1$ infinities of solutions to (3.9) on physical grounds. For example suppose the obstacles do not pierce the surface. Then it is reasonable to assume that as $ka \to \infty$, $r \to 0$, $t \to 1$ since very short waves will be unaffected. Thus

$$(r+t)/(r-t) = e^{2i(\theta_s-\theta_a)} \to -1, ka \to \infty$$

showing that $|\sin B| \to 1, ka \to \infty$, and

$$(r^2-t^2) = e^{2i(\theta_s+\theta_a)} \to -1, ka \to \infty$$

showing that $|\sin A| \to |\cos ka|, ka \to \infty$.

It is now clear that (3.9) has an infinity of solutions for ka for each m. Similar arguments can be used for surface-piercing obstacles. In the simplest case of two submerged obstacles it follows, by putting $n = 2, m = 1$ that $|R_2^{(i)}| = 0$ for $ka \sim \frac{1}{2}\pi(2s-1)$ (s integer) for large ka. In general each additional obstacle increases the set of solutions by one, and also changes the original set, unless members of the set $\{\cos m\pi/(n+1), m = 1, \ldots, n\}$ are contained in $\{\cos m\pi/n, m = 1, 2, \ldots, n-1\}$ in which case the corresponding solution set recurs.

We turn next to the sloshing problem for n identical bodies. Putting $A_0 = B_0$, $A_n = B_n$ as required by (2.11) gives

$$a_n + b_n = c_n + d_n,$$

whence from (2.39)

$$\frac{\sinh n\alpha}{\sinh \beta}(\cosh \beta + \cos D) = 0 \tag{3.10}$$

or

$$\frac{\sin n\alpha'}{\sinh \beta}(\cos A + \cos B \cos D)/\cos B = 0, \tag{3.11}$$

where we continue to assume $|\sin A/\sin B| < 1$ so that $\cos\alpha' = \sin A/\sin B$.

It follows that the resonant frequencies are given by solutions of the equations

$$\sin n\alpha' = 0 \quad (\beta \neq 0) \tag{3.12}$$

and

$$\cos A + \cos B \cos D = 0 \quad (\cos B \neq 0). \tag{3.13}$$

Equation (3.12) is precisely the requirement that $|R_n^{(i)}|$ vanish, a result noted by Heading (1982) in a similar context, whilst (3.13), being independent of n, must be identical to (2.35) for the resonant frequencies in any one of the intervals containing a single obstacle since these are clearly solutions also in the general n-obstacle case. The equivalence of (2.35) and (3.13) follows by noting that

$$\cos A \equiv \cos(kb+\theta_a)\cos(kc+\theta_s) - \sin(kb+\theta_a)\sin(kc+\theta_s)$$

and

$$\cos B \cos D \equiv \cos(\theta_s-\theta_a)\cos k(b-c) = \cos(kb+\theta_s)\cos(kc+\theta_a) \\ + \sin(kb+\theta_a)\sin(kc+\theta_s),$$

this latter result following from the identity

$$\cos(A+B)\cos(C+D) + \sin(A+D)\sin(B+C) = \cos(B-D)\cos(A-C)$$

true for all A, B, C, D. Notice that the solutions of (3.12) do not depend upon b, c but only upon $b+c = a$. Thus these resonant frequencies depend only on the spacing of the bodies and not their positions relative to the tank walls.

In order to test our results in this case we shall check against results provided by P. McIver (private communication) for two and three obstacles in the form of thin baffles.

It is known that for thin vertical obstacles $r+t=1$ and $\theta_s = \frac{1}{2}\pi$ so that $\sin A/\sin B = \cos(ka+\theta_a)/\cos\theta_a$ and condition (3.9) becomes

$$\cos ka - p\sin ka = \cos m\pi/n \quad (m = 1, 2, \ldots, n-1) \tag{3.14}$$

where

$$p = \tan\theta_a = \mathrm{i}r/t,$$

whilst (2.35) becomes

$$\cos(kc+\theta_a)\sin kb + \cos(kb+\theta_a)\sin kc = 0,$$

or

$$\sin ka = 2p\sin kb\sin kc, \tag{3.15}$$

in agreement with equation (3.7) of I.

Now for two baffles a distance $\frac{1}{2}a$ from the ends of a tank of length $2a$, McIver derives the conditions

$$\cos ka - \frac{2\mathrm{i}r}{1-r}\sin\tfrac{1}{2}ka\cos\tfrac{1}{2}\mathrm{ka} = 0$$

and

$$\sin ka - \frac{2\mathrm{i}r}{1-r}\sin^2\tfrac{1}{2}ka = 0.$$

The latter agrees with (3.15) when $b = c = \frac{1}{2}a$, whilst the former reduces to

$$\cos ka - p\sin ka = 0$$

in agreement with (3.14) with $n = 2, m = 1$.

A more convincing check is against the results for three identical equally spaced thin baffles a distance a apart, the outer two being a distance a from the tank walls. The results quoted in I (equation (3.8)) for the determination of the resonant frequencies is

$$\tan ka = \{2p \pm (2p^2+1)^{\frac{1}{2}}\}/(2p^2-1),$$

which arises from the solution of

$$(1-p^2)\cos 2ka - 2p\sin 2ka + p^2 = 0 \tag{3.16}$$

obtained after lengthy algebra following successive applications of the wide-spacing approximation (P. McIver, private communication). But (3.16) can be written

$$(1+\cos 2ka) + p^2(1-\cos 2ka) - 2p\sin 2ka - 1 = 0$$

or

$$\cos^2 ka + p^2\sin^2 ka - 2p\sin ka\cos ka = \tfrac{1}{2}$$

or

$$(\cos ka - p\sin ka) = \pm\frac{1}{\sqrt{2}}. \tag{3.17}$$

Also deduced by McIver is the result

$$\frac{\sin 2ka}{\sin^2 ka} = \frac{2\mathrm{i}r}{1-r} = 2p, \tag{3.18}$$

which can be interpreted as the solution that is symmetric about the middle baffle and can be deduced from (3.15) by putting $a = 2a, b = c = a$.

The general formula gives all these results directly. To achieve the same spacing a between the tank walls and the outer baffles as that between adjacent baffles we choose $n = 4$ and $b = 0$ (or a), $c = a$ (or 0) so that one of the (four) baffles coincides with one wall. Although this would appear to violate the wide-spacing approximation, it is justified since (3.12) holds for *any* value of b, c, with $b+c = a$.

The required condition is now, from (3.14)

$$\cos ka - p \sin ka = \cos \tfrac{1}{4} m\pi \quad (m = 1, 2, 3).$$

Clearly $m = 1, 3$ corresponds to (3.17) and $m = 2$ to (3.18).

4. Conclusion

The wide-spacing approximation has been used to consider the scattering of waves from an arbitrary number of identical bodies and to determine resonant frequencies when the same n bodies are bounded by the sides of a rectangular tank. A real-valued expression has been derived for the determination of the resonant frequencies when there is just a single obstacle and this has been generalized to n symmetric obstacles by utilizing a compact expression for the nth power of a 2×2 matrix. Expressions have also been derived, similar to those derived by Heading (1982), for the reflection and transmission coefficients for scattering of an incident wave by n symmetric obstacles.

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